DIVIDING POLYNOMIALS

Dividing polynomials by monomials

Ex: $\left(4x^4 - 8x^3 + 16x^2 - 12x\right):\left(4x\right)$

What is this?... What's the dividend? …divisor?...quotient?

Ex: $\left(15a^4 - 20a^2 + 10a^5 - 5a^3\right):\left(-5a^2\right)$

$\left(12a^4b - 2a^2b^6 + 9a^5b^2 - 5a^3b^3\right):\left(-3a^2b\right)$

$\left(\frac{4}{3}x^3y^5z - 2x^2y^4 - \frac{5}{4}x^2y^6z^3\right):\left(\frac{1}{6}xy^4\right)$

To divide a polynomial by a monomial, just divide each term in the polynomial by the monomial (by the right distributive property of division over addition/subtraction).
Division is right distributive over (with respect to) addition/subtraction
(Right distributive property holds for division over addition/subtraction)

$\left(-5x^4 - 8x^3 + 16x^2\right):\left(-2x^3\right)$ The polynomial isn't divisible by the monomial.

A polynomial is divisible by a monomial if each term of the polynomial is divisible by the monomial.

Steps for dividing a polynomial by a monomial: ***(Regents Prep)***

1. Divide **each** term of the polynomial by the monomial.
   a) Divide numbers (coefficients)
   b) Subtract exponents
   Keep this in mind: The number of terms in the polynomial equals the number of terms in the answer when dividing by a monomial.
2. Remember that numbers do not cancel and disappear! A number divided by itself is **1**.
3. Remember to write the appropriate sign in between the terms.

Dividing polynomials by polynomials

$(3x^3-2x^2+4x-3):(x^2+3x+3)=3x-11$

$28x+30$

The expression 3x-11is called the **quotient**, the expression x^2+3x+3 is called the **divisor** and the term 28x+30 is called the **remainder**. The remainder 28*x*+30 has degree 1, and is thus less than the degree of the divisor x^2+3x+3.

Polynomial long division

If you're dividing a polynomial by something more complicated than just a simple monomial, then you'll need a different method for simplification. The method is called "long division", is very like the long division of numbers you did back in elementary school, except that now you're dividing with variables. ***(Purplemath)***

Polynomial long division is a method used to divide polynomials.

Polynomial long division is essentially the same as long division for numbers. ***(Mathwords)***

12750	31
-124	411
35	
-31	
40	
-31	
9	

$$12750=31\cdot 411+9$$

$$\frac{12750}{31}=411+\frac{9}{31}$$

28537	19
-19	1501
95	
-95	
037	
-19	
18	

$$28537=19\cdot 1501+18$$

$$\frac{28537}{19}=1501+\frac{18}{19}$$

(The symbol separating the dividend from the divisor seems to have no established name, so can be simply referred to as the long division symbol.) ***(Wolfram MathWorld)***

Ex1 Divide $4x^2 + 18x + 8$ by $4x + 2$

First be sure that the terms of each polynomial are arranged in descending order.

Divide the leading term $4x^2$ of the dividend by the leading term 4x of the divisor, and write the answer x on the right side.

Now multiply this term x by the divisor 4x+2, and write the answer $-4x^2-2x$ under the first polynomial, changing all the signs and lining up terms of equal degree.

Draw the equals bar and add down. The first term will cancel out.

Remember to bring down the next term from the dividend.

Now repeat the procedure: Divide the leading term 16x of the polynomial on the last line by the leading term 4*x* of the divisor to obtain 4, and add this term to the x on the right side (write it next to the x).

Then multiply "back" changing all the signs -16x-8 and write the answer under the last line polynomial, lining up terms of equal degree.

Draw the equals bar and add down. Quotient is x+4 and remainder is zero.

Ex2 Divide $x^2 + 3x^3 - 5$ by $4 + x$

First, arrange (change) both polynomial in (to) descending order of the exponents of the variable. ($3x^3 + x^2 - 5$, $x + 4$)

Also, when one of the terms in the dividend is missing, leave space (leave proper room between terms) or insert 0 as the coefficient of the variable. (We write (insert) 0x since there is no "x" term). You can do this by turning the dividend into $3x^3 + x^2 + 0x - 5$. Then do the division.

Divide $3x^3$ by x (divide x into $3x^3$) and put this on the right side as the first part of the answer.

Multiply $3x^2$ by the divisor x+4 and place the products in the columns with like terms, changing all the signs.

Draw the equals bar and add down. The first term will cancel out.

Remember to carry (bring) down the next term from the dividend and start the procedure (process) again… You have to repeat the procedure one more time.

Multiply 44 times (x+4). …

(R=-181). At this point there is no term to bring down. You are done!

(In the next step, you would divide -181 by *x*, not getting a polynomial expression!)

The remainder is the last line: -181, and the quotient is the expression on the right: $3x^2 - 11x + 44$.

Ex3 Divide $a^3 - a^2 - 2a + 10$ by $a + 2$ (Q(a)= a^2-3 a +4; R(a)=2)

Ex4 Divide $5y^2 - 3y^3 + 4y^5 - 1$ by $2y^2 - y + 3$ (Q(y)=2y^3+y^2-4y-1; R(y)=11y+2)

Ex5 Divide $7x + 2x^2 + 3x^4 - 2$ by $x^2 - 2x + 2$ (Q(x)=3x^2+6x+8; R(x)=11x-18)

Since **12** : **3** = **4**, then **4** × **3** = **12**. ***Purplemath)***

If a division doesn't come out even, what am I supposed to do with the remainder? Think back to when you did long division with plain numbers. Sometimes there would be a remainder; for instance, if you divide **13** by **5** there is a remainder of **2**. Remember how you handled that?

If you get a remainder, you do the multiplication and then add the remainder back in. For instance, since **13** : **5** = **2** R **3**, then **13** = **5** × **2** + **3**. This process works the same way with polynomials. That is:

If A(x) : B(x) = Q(x) with remainder R(x), then A(x)=B(x) Q(x)+R(x).

In terms of our concrete example:

Since $(x^3 - 7x - 6):(x - 4) = x^2 + 4x + 9$ with remainder 30,

then $x^3 - 7x - 6 = (x - 4)(x^2 + 4x + 9) + 30$

For any polynomials A(x) and B(x), you can always find two polynomials Q(x) and R(x) such that

A(x)= B(x) Q(x) +R(x) or $\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)}$

where the degree of R(x) is less than the degree of B(x) *(Penguin+Re Fraschini)*

If deg(A(x))=m and deg(B(x))=n, $m \geq n$, then deg(Q(x))=m-n and deg(R(x))<n

A(x) *(read: A of x)* just means that the variable in the polynomial A is x.

In the special case where *R*(*x*)=0, we say that *B*(*x*) **divides evenly** into *A*(*x*).

Divisions in which there is zero (no) remainder are called **exact divisions**.

A so-called exact division is when the dividend is known to be an exact multiple of the divisor.

A division problem can be expressed in general form:

Dividend=(Divisor)(Quotient) + Remainder or $\frac{Dividend}{Divisor} = Quotient + \frac{\text{Re}mainder}{Divisor}$

We use this general form to check a division problem (how can you check your answer?).

We can verify the quotient by multiplying it by the divisor and adding the remainder to the result. We should end up with the original dividend. (***Karlscalculus***)

Ex2 Check: Dividend=(Divisor)(Quotient) + Remainder

$$x^3 - x^2 - 2x + 10 \stackrel{?}{=} (x+2)(x^2 - 3x + 4) + 2$$

$$x^3 - x^2 - 2x + 10 \stackrel{?}{=} x^3 - 3x^2 + 4x + 2x^2 - 6x + 8 + 2$$

$$x^3 - x^2 - 2x + 10 \stackrel{?}{=} x^3 - x^2 - 2x + 10$$

Yes √

Since the remainder is zero (that is, since there isn't anything left over), the division came out "even". When you do regular division with numbers and the division comes out even, it means that the number you divided by is a factor of the number you're dividing (for instance, if you divide 50 by 10 it will come out even, because 10 is a factor of 50). In this case, you now know that $x + 1$ is a factor of $x^2 - 9x - 10$, because the division **($x^2 - 9x - 10$) : ($x + 1$)** came out even. ***(Purplemath)***

Example *(Math Mutt)*

Divide $x^3 - 3x^2 + x + 8$ by x-2

$$
\begin{array}{r|l}
 & x^2 - x - 1 \\
x-2 & x^3 - 3x^2 + x + 8 \\
 & \underline{x^3 - 2x^2} \\
 & -x^2 + x \\
 & \underline{-x^2 + 2x} \\
 & -x + 8 \\
 & \underline{-x + 2} \\
 & 6
\end{array}
$$

Quotient is $x^2 - x - 1$

Remainder is 6

So $\dfrac{x^3 - 3x^2 + x + 8}{(x-2)} = (x^2 - x - 1) + \dfrac{6}{(x-2)}$

$$\Rightarrow x^3 - 3x^2 + x + 8 = (x-2)(x^2 - x - 1) + 6$$

Ruffini's rule or Synthetic Division ***(Purplemath, Wikipedia)***

In mathematics, **Ruffini’s rule** is a shortcut method (a simplified method) for (of) dividing (to divide) any polynomial by a linear (first degree) binomial of the form *x*–*c*, where *c* is a given constant (and *only* works in this case), that is it can be used only when the divisor is a linear binomial of the form *x*–*c*.

(The method **cannot** be used for non-linear divisors. If the dividing polynomial has degree 2 or more then we have to return to using the long division algorithm.)

It was described by Paolo Ruffini in 1809. Ruffini's rule is also known as **synthetic division.**

Ex1 Divide $(2x-11+3x^3)$ by $(x-3)$ ***(CliffsQuickRewiew, AlgebraII; Wikipedia)***

First, this problem will be done in the traditional manner.
Then it will be done by using the synthetic division method.

In the traditional manner …the division answer is: $Q(x)=3x^2+9x+29$, $R=76$

To do the problem using synthetic division, follow this procedure (I will display the process step-by-step):

1. Write the polynomial being divided in descending order. Then write only its coefficients (Take the coefficients of *A*(*x*) and write them down in order), using 0 for any missing terms (remember to put zeroes in for the powers of *x* that are not included in the polynomial; if you forget to leave "gaps", your division will not work properly!)
   Make sure you leave room underneath to write another row of numbers later.
2. Write the opposite of the constant term of the divisor at the left, just over the line.
   [Write (Put) the constant, *c*, of the divisor, x-c, at the left (at the bottom left edge, just over the line). In this problem, c=3].
3. Bring (Carry) down the first coefficient (the first number, representing the leading coefficient), just under the line.
4. Multiply the first coefficient by *c* (the number on the left). Then write this product (carry the result) over the line under the second coefficient (into the next column).
5. Add the second coefficient with the product and write the sum as shown. (Add the two values you’ve just put in the same column) (Add down the column)

6. Multiply by the number on the left and carry the result into the next column.
7. Add down the column.
8. Continue this process of multiplying and adding until there is a sum for the last column (until you've run out of numbers) (until you've finished). The completed division is …(*schema*)

The numbers along the bottom row are the coefficients of the quotient (result polynomial) with the powers of x in descending order. The last coefficient (the final value obtained) is the remainder.

The degree of the polynomial quotient in one less than that of A(x) the polynomial that was being divided. (The first power is one less than the highest power of the polynomial that was being divided)

Example ***(Maths Mutt)***

Divide $x^3 - 3x^2 + x + 8$ by x-2

	x^3	x^2	x	
2	1	-3	1	8
		2	-2	-2
	1	-1	-1	6

Divisor is x-2 Coefficients of Quotient Remainder = 6

x^2-x-1

$x^3 - 3x^2 + x + 8 = (x - 2)(x^2 - x - 1) + 6$

Example

Divide $3x^3 - 7x^2 + 5x + 4$ by $x + 3$

	x^3	x^2	x	
-3	3	-7	5	4
		-9	48	-159
	3	-16	53	-155

$3x^3 - 7x^2 + 5x + 4 = (x + 3)(3x^2 - 16x + 53) - 155$

And P(-3) = -155

Division by ax -c

If the divisor is a linear polynomial, ax-c, where a is not zero, the synthetic division method can be still be adapted to find the remainder. Clearly the case of a=1 is what we have already discussed.

As an example, let's work the problem where the dividend is $4x^3 + 6x^2 - 2x - 1$ and the divisor is 2x + 1. (***Internet***)

We learnt that: The quotient does not change if we multiply or divide the dividend and divisor by the same non-zero number.

Then we can divide both the dividend and the divisor by 2 and we obtain …

$$\left(2x^3 + 3x^2 - x - \frac{1}{2}\right) : \left(x + \frac{1}{2}\right)$$

Now we can use Ruffini's rule to do the division. …

$Q(x) = 2x^2 + 2x - 2$ $\quad R_1(x) = \frac{1}{2}$ $\quad R(x) = \frac{1}{2} \cdot 2$ (we have to double the remainder)

Hence the quotient is the same on dividing A(x) by *x-c/a* or *ax-c*. What changes is the remainder. To find the original remainder R(x) we must multiply the modified remainder, $R_1(x)$ ($R_1(x)$ is the remainder on dividing A(x) by *x-c/a*) by *a*. …

Quotient is $(2t^2 + 2t - 2)$, remainder is 1.

Example

Find the quotient and remainder when x^3+x^2+1 is divided by 4x-1

The remainder is $\frac{69}{64}$ and the quotient is $\frac{1}{4}x^2 + \frac{5}{16}x + \frac{54}{64}$

Dividing polynomials that have more than one variable (***Karlscalculus*** **+io)**

The polynomials in the examples above had the single variable, x. But there is a wider class of polynomials that have more than one variable. I will show you how we can use polynomial long division and Ruffini's rule to divide such polynomials.

It is to divide $2a^4 + a^3b - ab^3 + 2b^4$ (polynomial in two variables) by $a^2 - ab + 2b^2$ *(Persano)*.

First we must decide which of the letters we consider variable and which constant. For instance let's consider a as a variable and b as a constant and write both polynomial in descending order with respect to a; then do the division.

… $Q(a) = 2a^2 + 3ab - b^2$; $R(a) = -8ab^3 + 4b^4$

Then consider b as a variable and a as a constant.

Write both polynomial in descending order with respect to b, and then do the division.

… $Q(b) = b^2 - \frac{1}{2}a^2$; $R(b) = \frac{1}{2}a^3b + \frac{5}{2}a^4$

Let's repeat the procedure for another division: $(2xy^2 - x^2y - x^3 - 1) : (x + y)$ *(Sasso)*

… $Q(x) = -x^2 + 2y^2$; $R(x) = -2y^3 - 1$

… $Q(y) = 2xy - 3x^2$; $R(x) = 2x^3 - 1$

In this case let's do the division again using Ruffini's rule. …

Quotient and remainder depend on the variable we choose, so they are generally different.

<u>If a polynomial A is divisible by a polynomial B, that is the remainder is zero, writing both polynomials once in descending order with respect to one letter and then in descending order with respect to the other, we get the same quotient.</u>

Ex. $(x^4 + 2ax^3 + 4a^2x^2 - 2a^3x - 5a^4) : (x^2 - a^2)$ $Q(x) = Q(a) = x^2 + 2ax + 5a^2$ $R(x) = R(a) = 0$

The Remainder Theorem ***(Penguin Dictionary, Purplemath)***

If a polynomial *P(x)* (deg(*P*) ≥1) is divided by a linear binomial of the form *x-c*, then the remainder of this division is *P(c)*, that is the value obtained by substituting *c* into x in the polynomial *P(x)*.

The instruction "evaluate the polynomial P(x) when x is replaced with 4" is written as "find P(4)"

Evaluating a polynomial P(x) at a given (c) value of *x* means finding P(c), that is replacing the variable with the value c.

The value of a polynomial is found by replacing each letter in the expression with a number.

Example

If $P(x) = 2x^3+3x^2-7$, find the value of the polynomial at x = -2

The value of

$P(-2) = 2(-2)^3+3(-2)^2-7 = 2 \times (-8) + 3\times4 -7 = -16 +12 -7 = - 11$ (***Maths Mutt***)

P(c)=R

Proof.

If *P(x): (x-c)= Q(x)* with remainder *R(x)*, then we know that: *P(x)=Q(x) (x-c)+R(x)*

We learnt that the degree of *R(x)* is less than the degree of the divisor; our divisor is *x-c*, that is a linear factor, that is a factor in which the degree on *x* is "1", then the remainder must be just a constant. That is, when we divide by "*x – c*", our remainder will just be some number.

P(x)=Q(x) (x-c)+R

Substituting $x = c$ into the equation we see that $R = P(c)$

(We can evaluate the polynomial at $x = c$ (that is plug in the given value for x)).

$P(c) = (c – c)Q(c) + R$

$= (0)Q(c) + R$

$= 0 + R$

$= R$

Ex. P(-2) is the remainder when $P(x)=3x^5-8x^2-x-5$ is divided by x+2.

By synthetic division, the remainder is -131.

The remainder theorem is satisfied since $P(-2)=3(-2)^5-8(-2)^2-(-2)-5=-131$ (***Mathwords***)

Plug in -2 for *x*, remembering to be careful with parentheses and negatives.

Always remember to be careful with the minus signs!

Ruffini’s Theorem *(CliffsQuickRewiew, AlgebraII, io)*

We know that:

- A polynomial *P(x)* is divisible by *x-c* if and only if the remainder of the division of *P(x)* by *x-c* is 0;
- When P(x) is divided by *x-c* the remainder is *P(c)* (Remainder Theorem).

Then:

Ruffini’s theorem:

A polynomial *P(x)* is divisible by *x-c* if and only if *P(c)=0*
In other terms:
A polynomial *P(x)* is divisible by *x-c* if and only if *c* is a zero of *P(x).*

Factor theorem: *x-c* is a factor of *P(x)* if and only if *P(c)=0*
(CliffsQuickRewiew, AlgebraII, Purplemath)

The Factor Theorem is a result of the Remainder Theorem, and is based on the same reasoning.
Remember that, if you divide a polynomial $p(x)$ by a factor $x - c$ of that polynomial, then you will get a zero remainder. Let's look again at that Division Algorithm expression of the polynomial:

$$P(x) = (x - c)Q(x) + R(x)$$

If $x - a$ is indeed a factor of $P(x)$, then the remainder after division by $x - c$ will be zero. That is:

$$P(x) = (x - c)Q(x)$$

In terms of the Remainder Theorem, this means that, if $x - c$ is a factor of $P(x)$, then the remainder, when we do synthetic division by $x = c$, will be zero.

The point of the Factor Theorem is that, if you synthetic-divide a polynomial by $x = c$ and get a zero remainder, then, not only is $x = c$ a zero of the polynomial (because of the Remainder Theorem), but $x - c$ is also a factor (because of the Factor Theorem).

Ex. Use the Factor Theorem to determine whether $x - 1$ is a factor of $2x^4 + 3x^2 - 5x + 7$.
… Since the remainder is not zero, then, according to the Factor Theorem: x – 1 is not a factor of the polynomial.